A Conditional Mutual Information Estimator for Mixed Data and an Associated Conditional Independence Test

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Outline

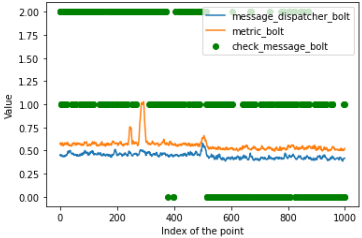
1. Challenges & Objectives
2. CMI Estimator for Mixed Data and Associated Test
3. Limitations and future work

Challenges & Objectives

- Mixed data occur frequently in many applications, such as health, marketing, medical, and finance. (Ahmad and Dey, 2007; Hennig and Liao, 2013; Morlini and Zani, 2010)

  e.g.

Index	message_dispatcher_bolt	metric_bolt	check_message_bolt
1	0.56	0.51	*Normal*
2	0.60	0.53	*Warning*
3	0.87	0.52	*Critial*
4	1.06	0.51	*Normal*
5	0.58	0.54	*Normal*
...	...	...	...



Challenges & Objectives

- Measuring the (in)dependence between random variables from data when the underlying joint distribution is unknown plays a key role in several settings:
    1. Causal discovery (Spirtes et al., 2000)
    2. Graphical model inference (Whittaker, 2009)
    3. Feature selection (Vinh, Chan, and Bailey, 2014)

- **Objectives: Estimating** and **testing** conditional independence via Conditional Mutual Information (CMI), from observable mixed data.

- Conditional Mutual Information (CMI) has good properties:

$$I(X, Y|Z) = 0 \Rightarrow X \perp\!\!\!\perp Y|Z$$
$$I(X, Y|Z) \neq 0 \Rightarrow X \not\perp\!\!\!\perp Y|Z$$

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Hybrid conditional mutual information estimation for mixed data (CMIh)

- Consider 3 mixed random vectors $X^{t,\ell}$, $Y^{t,\ell}$ and $Z^{t,\ell}$. $X^{t,\ell}$(resp. $Y^{t,\ell}$, $Z^{t,\ell}$) can be denoted as (X^t, X^ℓ), where
  - X^t contains all quantitative dimensions of $X^{t,\ell}$
  - X^ℓ contains all qualitative dimensions of $X^{t,\ell}$
- The Conditional Mutual Information $I(X^{t,\ell}; Y^{t,\ell}|Z^{t,\ell})$ is defined as:

$$I(X^{t,\ell}; Y^{t,\ell}|Z^{t,\ell}) = H(X^\ell, Z^\ell) + H(Y^\ell, Z^\ell) - H(X^\ell, Y^\ell, Z^\ell) - H(Z^\ell) + H(X^t, Z^t|X^t, Z^t)$$
$$+H(Y^t, Z^t|Y^\ell, Z^\ell) - H(X^t, Y^t, Z^t|X^\ell, Y^\ell, Z^\ell) - H(Z^t|Z^\ell)$$

$\Longrightarrow$ $I(X^{t,\ell}; Y^{t,\ell}|Z^{t,\ell})$ is a combination of

- Entropy of qualitative dimensions
- Entropy of quantitative dimensions conditioning on qualitative dimensions

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Hybrid conditional mutual information estimation for mixed data (CMIh)

- Entropy of qualitative dimensions can be estimated as:

$$\widehat{H}(X^\ell, Z^\ell) = -\sum_{\substack{x^\ell \in \Omega(X^\ell) \\ z^\ell \in \Omega(Z^\ell)}} \widehat{P}_{X^\ell, Z^\ell}(x^\ell, z^\ell) * \log\left(\widehat{P}_{X^\ell, Z^\ell}(x^\ell, z^\ell)\right)$$

- Entropy of quantitative dimensions conditioning on qualitative dimensions can be estimated as:

$$\widehat{H}(X^t, Z^t | X^\ell, Z^\ell) = \sum_{\substack{x^\ell \in \Omega(X^\ell) \\ z^\ell \in \Omega(Z^\ell)}} \widehat{P}_{X^\ell, Z^\ell}(x^\ell, z^\ell) \widehat{H}(X^t, Z^t | X^\ell = x^\ell, Z^\ell = z^\ell)$$

  - $\widehat{P}_{X^\ell, Z^\ell}(x^\ell, z^\ell)$ is estimated using histograme.
  - $\widehat{H}(X^t, Z^t | X^\ell = x^\ell, Z^\ell = z^\ell)$ is estimated using the nearest neighbors estimator (Kozachenko and Leonenko, 1987):

$$\widehat{H}(X^t, Z^t | X^\ell = x^\ell, Z^\ell = z^\ell) = \psi(n_{xz}) - \psi(k_{xz}) + \log\left(v_{d_{xz}}\right) + \frac{d_{xz}}{n_{xz}} \sum_{i=1}^{n_{xz}} \log \xi_{xz}(i)$$

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Hybrid conditional mutual information estimation for mixed data (CMIh)

Experiments of estimator

- Configuration of experiments:

Scenarios	X	Y	Z
Dependence quantitative	quantitaive, Gaussian	quantitative, Gaussian	
Dependence mixed	qualitative, uniform	quantitative, uniform	
Dependence mixed imbalanced	quantitative, exponential	qualitative, Poisson	
Conditional dependence quantitative	quantitaive, Gaussian	quantitative, Gaussian	qualitative, binomial
Conditional dependence mixed	qualitative, uniform	quantitative, uniform	qualitative, binomial
Conditional dependence mixed imbalanced	quantitative, exponential	qualitative, Poisson	qualitative, binomial
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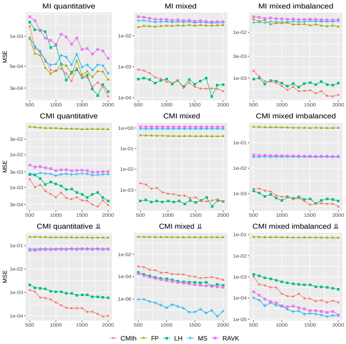
Hybrid conditional mutual information estimation for mixed data (CMIh)

Experiments of estimator

- MSE (on a log-scale) of each method with sample size $n \in \{500, 600, ..., 2000\}$ over 100 repetitions.



- Conclusions:
  - FP performs well in the purely quantitative case;
  - MS and RAVK have similar performance in most cases and MS has a main drawback as it gives the value close to 0 in some particular cases;
  - LH and CMIh, overall, are more robust than the other ones, but LH is more computation-costly than CMIh.

Local-Adaptive permutation test for mixed data (LocAT)

- Hypothesis Null & Hypothesis Alternative:
  - $H_0 : X \perp\!\!\!\perp Y | Z$
  - $H_1 : X \not\perp\!\!\!\perp Y | Z$
- The main concept of the local permutation test (LocT) can be described as follows for three one-dimensional random variables, namely X, Y, and Z:
  1. Estimate the conditional mutual information of the original data as $\hat{I}(X, Y|Z)$,
  2. Shuffle the value of X within its neighbours that have a similar Z value, resulting in X_π. This permutation ensures that $X_\pi \perp\!\!\!\perp Y|Z$,
  3. Repeat Step 2 B times, and estimate $\hat{I}_i(X\pi, Y|Z)$ for each permutation $i \in \{1, \dots, B\}$,
  4. Calculate p-value by using $\hat{I}(X, Y|Z)$ and $\{\hat{I}_i(X\pi, Y|Z)\}_{i\in\{1,\dots,B\}}$.

* Intuitive explanations:
  - If $X \perp\!\!\!\perp Y|Z$, in most cases, $\hat{I}_i(X_\pi, Y|Z) \approx \hat{I}(X, Y|Z)$, where $i \in \{1, \dots, B\}$.
  - If $X \not\perp\!\!\!\perp Y|Z$, in most cases, $\hat{I}_i(X_\pi, Y|Z) \geq \hat{I}(X, Y|Z)$, where $i \in \{1, \dots, B\}$.
- Extend local permutation test (LocT) to mixed data, by defining the nearest neighbours should have the same qualitative values in Z and denote it as (LocAT).

Local-Adaptive permutation test for mixed data (LocAT)

Experiments of independent test

- Here, we propose to analyze 3 structures that are classical:

  Chain: $X- > Z- > Y$
  Fork: $X < -Z- > Y$
  Collider: $X- > Z < -Y$

- For each structure, we consider the following configurations of experiments:

  $t\ell t$. X and Z are quantitative, Y is qualitative;
  ttt. X Y and Z are quantitative;
  $\ell\ell t$. X and Y are qualitative, Z is quantitative;
  $t\ell\ell$. X is quantitative, Y and Z are qualitative;
  $tt\ell$. X and Y are quantitative, Z is qualitative;
  $\ell\ell\ell$. X Y and Z are qualitative.

- We use acceptance rate over 10 repetitions of two threholds (0.01 and 0.05) to show the results:
  - The acceptance rate closer to 1 under different threshold the better.
  - The number of sampling point is 500.

Local-Adaptive permutation test for mixed data (LocAT)

Experiments of independent test

		CMIh-LocT		CMIh-LocAT		CMIh-GloT		MS-LocT		MS-LocAT		MS-GloT	
		0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
Chain	$t\ell t$	1	1	1	1	0	0	1	1	1	1	1	1
	ttt	1	1	1	1	0	0	1	0.9	1	0.9	0	0
	$\ell\ell t$	1	0.9	1	0.9	1	0.8	1	1	1	1	1	1
	$t\ell\ell$	1	1	1	1	1	1	1	1	1	1	1	1
	$tt\ell$	0	0	0.8	0.4	0	0	0	0	0.5	0.3	0	0
	$\ell\ell\ell$	1	0.9	1	0.9	1	1	1	1	1	1	1	1
Fork	$t\ell t$	0.9	0.9	0.9	0.9	0	0	1	1	1	1	1	1
	ttt	1	1	1	1	0	0	1	1	1	1	0	0
	$\ell\ell t$	1	1	1	1	1	1	1	1	1	1	1	1
	$t\ell\ell$	1	1	1	0.9	1	1	1	1	1	1	1	1
	$tt\ell$	0	0	0.9	0.8	0	0	0	0	0.8	0.5	0	0
	$\ell\ell\ell$	1	1	1	1	1	1	1	0.9	1	1	1	1
Collider	$t\ell t$	1	1	1	1	1	1	0	0	0	0	0	0
	ttt	1	1	1	1	0.8	0.9	1	1	1	1	1	1
	$\ell\ell t$	1	1	1	1	1	1	0	0	0	0	0	0
	$t\ell\ell$	0	0	0.4	0.7	0	0	0	0	0	0	0	0
	$tt\ell$	0.6	1	1	1	0.2	0.4	0	0	0	0	0	0
	$\ell\ell\ell$	1	1	1	1	1	1	1	1	1	1	0.4	0.9

- Conclusions:
  - CMIh with the test LocAT allows one to correctly identify the true (in)dependence relation on all configurations of all structures;
  - For other combinations, these is at least one case where it can not work.

Outline

Limitations and future work

- The proposed test may suffer from the problem indicated in [1].
- Check the performance of the method on more sophisticated structures.

[1] Shah, Rajen & Peters, Jonas. (2018). The Hardness of Conditional Independence Testing and the Generalised Covariance Measure. Annals of Statistics. 48. 10.1214/19-AOS1857.

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Thank you !